IITJEE - 2008 Mathematics Paper - I

PART - I (MATHEMATICS)

SECTION - I (Straight Objective Type)

1. If $0 < x < 1$, then $\sqrt{1+x^2}\left[\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2-1\right]^{\frac{1}{2}} =$

(A) $\frac{1}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

1. **(C)** Expr = $\sqrt{1+x^2}\left[\left(\frac{x^2}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right)^2-1\right] = x\sqrt{1+x^2}$.

2. Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$ then,
(A) C_1 and C_2 touch each other only at one point.
(B) C_1 and C_2 touch each other exactly at two points.
(C) C_1 and C_2 intersect (but do not touch) at exactly two points.
(D) C_1 and C_2 neither intersect nor touch each other.

2. **(B)** Solving both simultaneously $x^2 - 2x + 1 = 0 \Rightarrow x = 1$,
distance of centre of circle is from origin is less than radius of the circle i.e. $2\sqrt{2}$.
$\therefore$ both the circle and the parabola are symmetrical about x-axis, so they will touch each other exactly at two points.

3. The edges of a parallelopiped are of unit length and are parallel to non-coplanar unit vectors $\vec{a}, \vec{b}, \vec{c}$ such that $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 1/2$. Then, the volume of the parallelopiped is

3. **(A)** $V = [\vec{a}\ \vec{b}\ \vec{c}] = \sqrt{([a\ b\ c]^2)}$

$$= \sqrt{\begin{vmatrix} a.a & a.b & a.c \\ b.a & b.b & b.c \\ c.a & c.b & c.c \end{vmatrix}} = \sqrt{\begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix}} = 1/\sqrt{2}.$$

4. Let a and b be non zero real numbers. Then, the equation
$(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents

4. **(B)** $(ax^2 + by^2 + c)(x - 2y)(x - 3y) = 0$ represents two straight lines $x - 2y = 0$ and $x - 3y = 0$ and $ax^2 + by^2 + c = 0$ represents a circle if $a = b$ and c is of opposite sign as that of a.

5. The total number of local maxima and local minima of the function

$$f(x)=\begin{cases}(2+x)^3, & -3<x\le -1\\ x^{2/3}, & -1<x<2\end{cases} \text{ is}$$

5. **(C)** Local minima at (0, 0) and maxima at (– 1, 1).

6. Let $g(x)=\dfrac{(x-1)^n}{\log\cos^m(x-1)}$; $0 < x < 2$, m and n are integers, $m \neq 0$, $n > 0$ and let p be the left hand derivative of $|x-1|$ at $x = 1$. If $\lim\limits_{x\to 1^+} g(x)=p$, then

6. **(C)** $\lim\limits_{h\to 0}\dfrac{(1+h-1)^n}{(m/n)\dfrac{\cos(1+h-1)}{(\cos h-1)}(\cos h-1)}=-1$

$\Rightarrow$ $h^n / m\, 2\sin^2(h/2) = 1$ $\Rightarrow$ $h = 2, m = 2$.

SECTION - II (Multiple Correct Answers Type)

7. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equation of parabolas with latus rectum PQ are

7. **(B)(C)** End points of latus rectum of parabola are $(-\sqrt{3}, -1/2)$ and $(\sqrt{3}, -1/2)$.
So, vertex of parabola will be $(0, -1/2)$.
Directrix of parabola will be $y = \sqrt{3} - (1/2)$ or $y = -\sqrt{3} - (1/2)$.
Hence, equation of parabola is $x^2 + 2\sqrt{3}\, y = 3 - \sqrt{3}$ or $x^2 - 3\sqrt{3}\, y = 3 + \sqrt{3}$.

8. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then

8. **(B)(D)** $PS\,.\,ST = QS\,.\,SR$
Using G.M. > H.M. for unequal quantity.
$\sqrt{(PS\,.\,ST)} > 2/((1/PS) + (1/ST))$
$\Rightarrow$ $\sqrt{(QS\,.\,SR)} > 2/((1/PS) + (1/ST))$ **....** (1)
also A.M. > H.M. for unequal quantity.
$(SR + QS)/2 > \sqrt{(QS\,.\,SR)} > 2/((1/PS) + (1/ST))$ from (1)
$\Rightarrow$ $(1/PS) + (1/ST) > 1/QR$.

9. Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'(1/4) = 0$. Then

9. **(A)(B)(C)(D)** $f'(1/4) = 0$, $f'(x) + f'(1-x) = 0$
$\Rightarrow$ $f'(3/4) = 0$ and $f'(1/2) = 0$
$\Rightarrow$ $f''(C_1) = 0$, $1/2 < C_1 < 3/4$ and $f''(C_2) = 0$, $0 < C_2 < 1/2$
$I = \int_{-1/2}^{1/2} f(x + (1/2)) \sin x\, dx = \int_{-1/2}^{1/2} f((1/2) - x) \sin x\, dx = -I$
{as $f((1/2) - x) = f((1/2) + x)$} $\Rightarrow$ $I = 0$
$\int_0^{1/2} f(t)\, e^{\sin \pi t}\, dt = \int_1^{1/2} f(1-z)\, e^{\sin(\pi - \pi z)} (-dz)$, $z = 1 - t$
$= \int_1^{1/2} f(1-z)\, e^{\sin \pi z}\, dz$.

10. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$, for n = 1, 2, 3, ... Then

10. **(A)(D)** Trapezoidal area for T_n is higher whereas for S_n it is lower.

SECTION - III (Reasoning Type)

11. Consider the system of equations

$x - 2y + 3z = -1$

$-x + y - 2z = k$

$x - 3y + 4z = 1.$

STATEMENT - 1 **:** The system of equations has no solution for $k \neq 3$.

and

STATEMENT - 2 **:** The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, $k \neq 3$.

(A) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1.

(B) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a not a correct explanation for STATEMENT - 1.

(C) STATEMENT - 1 is True, STATEMENT - 2 is False.

(D) STATEMENT - 1 is False, STATEMENT - 2 is True.

11. **(A)** $x / D_x = y / D_y = z / D_z = 1 / D$

Here $D = 0$, so if $D_y \neq 0$, the system has no solution.

$$\therefore \begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} = 3 - k \neq 0 \text{ for } k \neq 3.$$

12. Consider the system of equations

$ax + by = 0$, $cx + dy = 0$, where $a, b, c, d \in \{0, 1\}$.

STATEMENT - 1 **:** The probability that the system of equations has a unique solution is 3 / 8.

and

STATEMENT - 2 **:** The probability that the system of equations has a solution is 1.

(A) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1.

(B) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a not a correct explanation for STATEMENT - 1.

(C) STATEMENT - 1 is True, STATEMENT - 2 is False.

(D) STATEMENT - 1 is False, STATEMENT - 2 is True.

12. **(B)** a, b, c, d can be 0 or 1.

$ad \neq bc$ for unique solution, which is not possible in following ways.

(i) any three of them are zero $\rightarrow$ 4 ways.

(ii) all of them should be zero or one $\rightarrow$ 2 ways.

(iii) one among a, d is zero and one among b, c is zero $\rightarrow$ 4 ways.

Total no. of ways = 16 $\quad \therefore \quad p = 1 - (10 / 16) = 3 / 8$.

trivial solution clearly exists.

13. Let f and g be real valued functions defined on interval $\{-1, 1\}$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$ and $f(x) = g(x) \sin x$.

STATEMENT - 1 **:** $\lim\limits_{x \to 0} [g(x) \cot x - g(0) \operatorname{cosec} x] = f''(0)$.

and

STATEMENT - 2 **:** $f'(0) = g(0)$.

(A) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1.

(B) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a not a correct explanation for STATEMENT - 1.

(C) STATEMENT - 1 is True, STATEMENT - 2 is False.

(D) STATEMENT - 1 is False, STATEMENT - 2 is True.

13. **(B)** $\lim\limits_{x \to 0} \dfrac{g(x) \cos x - g(0)}{\sin x} = \lim\limits_{x \to 0} \dfrac{g'(x) \cos x - \sin x \, g(x)}{\cos x} = g'(0) = 0$

$f'(x) = g'(x) \sin x + \cos x \, g(x)$

$f''(x) = g''(x) \sin x + g'(x) \cos x + \cos x \, g'(x) - \sin x \, g(x)$

$f''(0) = 0$.

14. Consider three planes

$P_1 : x - y + z = 1$

$P_2 : x + y - z = -1$

$P_3 : x - 3y + 3z = 2$.

Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, and P_1 and P_2, respectively.

STATEMENT - 1 **:** At least two of the lines L_1, L_2 and L_3 are non parallel.

and

STATEMENT - 2 **:** The three planes do not have a common point.

(A) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1.

(B) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a not a correct explanation for STATEMENT - 1.

(C) STATEMENT - 1 is True, STATEMENT - 2 is False.

(D) STATEMENT - 1 is False, STATEMENT - 2 is True.

14. **(D)** L_1 is $z = y + (3 / 4)$, $x = -1 / 4$
L_2 is $z = y + (1 / 2)$, $x = 1 / 2$
L_3 is $z = y + 1$, $x = 0$.

SECTION - IV (Linked Comprehension Type)

Paragraph for Question Nos. 15 to 17

Let A, B and C be three sets of complex numbers as defined below
$A = \{z : \text{Im}\, z \geq 1\}$
$B = \{z : |z - 2 - i| = 3\}$
$C = \{z : \text{Re}\,(1 - i)\, z = \sqrt{2}\}$.

15. The number of elements in the set $A \cap B \cap C$ is

15. **(B)** Intersection of $x + y = \sqrt{2}$ and $y = 1$ is $(\sqrt{2} - 1, 1)$ lying within the circle.

16. Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

16. **(C)** $(\text{diameter})^2 = 6^2 = 36$.

17. Let z be any point in $A \cap B \cap C$ and let ω be any point satisfying $|\omega - 2 - i| < 3$. Then $|z| - |\omega| + 3$ lies between

17. **(B)** $|z| + 3$ is constant $\therefore \quad 0 < |\omega| < \sqrt{5} + 3$
$\Rightarrow \quad |z| - \sqrt{5} < |z| - |\omega| + 3 < |z| + 3$
$\therefore \quad |z| \approx \sqrt{5}$
Hence $0 <$ Exp. < 6
(C) and (D) are also correct.

Paragraph for Question Nos. 18 to 20

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact C with the sides PQ, QR, RP are D, E, F, respectively. The line PQ is given by the equation $\sqrt{3}\, x + y - 6 = 0$ and the point D is $(3\sqrt{3} / 2, 3 / 2)$. Further, it is called given that the origin and the centre of C are on the same side of the line PQ.

18. The equation of circle C is

18. **(D)** Let (h, k) = centre $\quad \Rightarrow \quad \sqrt{3}\, h + k - 6 < 0$
also $|(\sqrt{3}\, h + k - 6) / 2| = 1 \quad \Rightarrow \quad \sqrt{3}\, h + k = 4$
$\therefore \quad (k - (3 / 2)) / (h - (3\sqrt{3} / 2)) = 1 / \sqrt{3}$

$\Rightarrow \quad C \equiv (\sqrt{3}, 1)$

Circle **:** $(x - \sqrt{3})^2 + (y - 1)^2 = 1 \quad \because \quad CR = 2.$

$RC / RD = (2 / 3)$ and R is $(0, 0)$.

19. Points E and F are given by

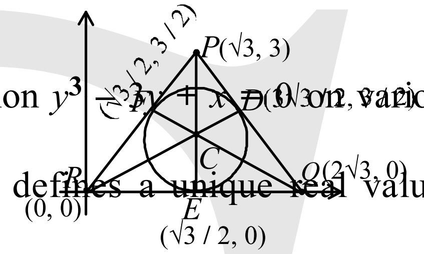
19. **(A)** E **:** $(\cot 30°, 0) \equiv (\sqrt{3}, 0)$

F **:** $(RF \cos 60°, RF \sin 60°) \equiv (\sqrt{3} / 2 , 3 / 2)$.

20. Equations of the sides QR, RP are

20. **(D)** QR **:** $y = 0$

RP **:** $y = \tan 60° \,.\, x = \sqrt{3}\, x$.



Paragraph for Question Nos. 21 to 23

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line.

If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$.

If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

21. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$

21. **(B)** $x = 3y - y^3$

$dx / dy = 3 - 3y^2$

$d^2y / dx^2 = (d / dx)\,(1 / 3(1 - y^2)) = d(1 / 3(1 - y^2)) / d(3y - y^3)$

$= -(1 - y^2)^{-2}\,(-2y) / 3(3 - 3y^2) = 2y / (1 - y^2)^3\, 3^2$

$= 4\sqrt{2} / (-7^3 \,.\, 3^2)$.

22. The area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

22. **(A)** $f'(x) = 1 / 3(1 - (f(x))^2)$

$$\int_a^b f(x)dx = x\, f(x)\Big|_a^b - \int_a^b f'(x)\, x\; dx.$$

23. $\displaystyle\int_{-1}^{1} g'(x)dx =$

23. **(D)** $\int_{-1}^{1} g'(x)\, dx = g(1) - g(-1) = 2 - (-2) = 4.$