IITJEE - 2008 Mathematics Paper - II

PART - I (MATHEMATICS)

SECTION - I (Straight Objective Type)

1. Let $g(x) = \log f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x + 1) = x\,f(x)$. Then, for $N = 1, 2, 3, \ldots\ldots$,

$$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=$$

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots\ldots+\frac{1}{(2N-1)^2}\right\}$ (B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots\ldots+\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots\ldots+\frac{1}{(2N+1)^2}\right\}$ (D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots\ldots+\frac{1}{(2N+1)^2}\right\}$

1. **(A)** $g(x + 1) = \log x + g(x)$
$g'(x + 1) = (1 / x) + g'(x)$
$g''(x + 1) = -(1 / x^2) + g''(x)$
$g''(N + (1 / 2)) - g''(1 / 2)$
$= (g''(1 + (1 / 2)) - g''(1 / 2)) + (g''(2 + (1 / 2)) - g''(1 + (1 / 2))$
$+ \ldots. + (g''(N + (1 / 2)) - g''(N - (1 / 2))$

$$= -4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots\ldots+\frac{1}{(2N-1)^2}\right\}$$

2. Let $I=\int\frac{e^x}{e^{4x}+e^{2x}+1}dx,\ J=\int\frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$.

Then, for an arbitrary constant C, the value of $J - I$ equals

(A) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+C$ (B) $\frac{1}{2}\log\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+C$

(C) $\frac{1}{2}\log\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+C$ (D) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+C$

2. **(C)** By $e^x + e^{-x} = t$,
$J - I = \int dt / (t^2 - 1) = (1 / 2)\, ln\, |\, (t - 1) / (t + 1)\, | + c.$

3. Let two non collinear unit vectors $\vec{a}$ and $\vec{b}$ form an acute angle. A point P moves so that at any time t the position vector $\overrightarrow{OP}$ (where O is the origin) is given by $\vec{a}\cos t + \vec{b}\sin t$. When P is farthest from origin O, let M be the length of $\overrightarrow{OP}$ and $\vec{u}$ be the unit vector along $\overrightarrow{OP}$. Then,

(A) $\vec{u} = \dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$ and $M = (1 + \vec{a}\cdot\vec{b})^{1/2}$

(B) $\vec{u} = \dfrac{\vec{a}-\vec{b}}{|\vec{a}-\vec{b}|}$ and $M = (1 + \vec{a}\cdot\vec{b})^{1/2}$

(C) $\vec{u} = \dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$ and $M = (1 + 2\vec{a}\cdot\vec{b})^{1/2}$

(D) $\vec{u} = \dfrac{\vec{a}-\vec{b}}{|\vec{a}-\vec{b}|}$ and $M = (1 + 2\vec{a}\cdot\vec{b})^{1/2}$

3. **(A)** Furthest when $\sin t = \cos t = 1/\sqrt{2}$

$\therefore \quad \vec{u} = (\vec{a} + \vec{b}) / |\vec{a} + \vec{b}|$.

Also, $M = \sqrt{2}\cos(\theta/2) = (1 + \vec{a}\cdot\vec{b})^{1/2}$.

4. Consider a branch of the hyperbola
$x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$
with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is

(A) $1 - \sqrt{\dfrac{2}{3}}$ (B) $\sqrt{\dfrac{3}{2}} - 1$ (C) $1 + \sqrt{\dfrac{2}{3}}$ (D) $\sqrt{\dfrac{3}{2}} + 1$

4. **(B)** $\dfrac{(x-\sqrt{2})^2}{4} - \dfrac{(y+\sqrt{2})^2}{2} = 1 \quad \Rightarrow \quad a = 2,\ b = \sqrt{2},\ e = \sqrt{(1.5)}$

$\Delta = (1/2)\,a\,(e-1)\,.\,(b^2/a) = e - 1 = \sqrt{(1.5)} - 1$.

5. Let the function $g : (-\infty, \infty) \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \dfrac{\pi}{2}$. Then, g is

(A) even and is strictly increasing in $(0, \infty)$
(B) odd and is strictly decreasing in $(-\infty, \infty)$
(C) odd and is strictly increasing in $(-\infty, \infty)$
(D) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

5. **(C)** $g(-u) = 2\cot^{-1}(e^u) - (\pi/2)$
$g(u) = 2\tan^{-1}(e^u) - (\pi/2) \qquad \therefore \quad g(u) + g(-u) = 0.$

6. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\vec{i} + \vec{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by

(A) $6 + 7i$ (B) $-7 + 6i$ (C) $7 + 6i$ (D) $-6 + 7i$

6. **(D)** $z_1 = 6 + 5i$, $z_{11} = 7 + 6i$, $z_2 = i\, z_{11} = 7i - 6$.

7. An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, the number of outcomes that B must have so that A and B are independent, is

(A) 2, 4 or 8 (B) 3, 6 or 9 (C) 4 or 8 (D) 5 or 10

7. **(D)** $P(AB) = P(A)\,.\,P(B)$

$m / 10 = (4 / 10)\,.\,(n / 10)$

$\Rightarrow \quad 10\, m = 4\, n \quad \Rightarrow \quad 5\, m = 2\, n$

8. Consider three points $P = (-\sin(\beta - \alpha), -\cos\beta)$, $Q = (\cos(\beta - \alpha), \sin\beta)$ and $R = (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$, where $0 < \alpha, \beta, \theta < \pi / 4$. Then

(A) P lies on the line segment RQ (B) Q lies on the line segment PR

(C) R lies on the line segment QP (D) P, Q, R are non collinear

8. **(D)** $\begin{vmatrix} -\sin(\beta-\alpha) & -\cos\beta & 1 \\ \cos(\beta-\alpha) & \sin\beta & 1 \\ \cos(\beta-\alpha+\theta) & \sin(\beta-\theta) & 1 \end{vmatrix} \neq 0$.

9. The area of the region between the curves $y = \sqrt{\frac{1+\sin x}{\cos x}}$ and $y = \sqrt{\frac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \pi / 4$ is

(A) $\displaystyle\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$ (B) $\displaystyle\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$

(C) $\displaystyle\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$ (D) $\displaystyle\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$

9. **(B)** By $t = \tan(x / 2)$, $y_1 = \sqrt{\frac{1+t}{1-t}}, y_2 = \sqrt{\frac{1-t}{1+t}}$ $\quad \therefore \quad I = \displaystyle\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$.

SECTION - II (Reasoning Type)

10. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and α, $1 / \beta$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$.

    STATEMENT - 1 **:** $(p^2 - q)(b^2 - ac) \geq 0$,

    and

    STATEMENT - 2 **:** $b \neq pa$ or $c \neq qa$.

    (A) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1.
    (B) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a not a correct explanation for STATEMENT - 1.
    (C) STATEMENT - 1 is True, STATEMENT - 2 is False.
    (D) STATEMENT - 1 is False, STATEMENT - 2 is True.

10. **(B)** Roots are not complex conjugate
    Since $\beta^2 \notin \{-1, 0, 1\}$
    $\therefore \quad p^2 - q \geq 0$ and $b^2 - ac \geq 0$
    Also, $\beta^2 = qa / c$ and $(\beta^2 - 1) / \beta = 2(b - ap) / a$.

11. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

    STATEMENT - 1 **:** The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.

    and

    STATEMENT - 2 **:** The numbers b_1, b_2, b_3, b_4 are in H.P.

    (A) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1.
    (B) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a not a correct explanation for STATEMENT - 1.
    (C) STATEMENT - 1 is True, STATEMENT - 2 is False.
    (D) STATEMENT - 1 is False, STATEMENT - 2 is True.

11. **(C)** b_k's are sum of G.P. i.e. $b_k = a(1 - r^k) / (1 - r)$
    which are neither in A.P., G.P. or H.P.

12. Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2 - 1}\, dy - y\sqrt{y^2 - 1}\, dx = 0$ satisfy $y(2) = 2 / \sqrt{3}$.

STATEMENT - 1 **:** $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$.

and

STATEMENT - 2 **:** $y(x)$ is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$.

(A) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1.
(B) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a not a correct explanation for STATEMENT - 1.
(C) STATEMENT - 1 is True, STATEMENT - 2 is False.
(D) STATEMENT - 1 is False, STATEMENT - 2 is True.

12. **(C)** Solving $\sec^{-1} y - \sec^{-1} x = c = -\pi / 6$

$$\Rightarrow \quad \frac{1}{y} = \frac{1}{\frac{\sqrt{3}}{2x} + \frac{1}{2}\sqrt{1 - \frac{1}{x^2}}}.$$

13. Consider
$L_1 : 2x + 3y + p - 3 = 0$
$L_2 : 2x + 3y + p + 3 = 0$,
where p is a real number, and $C : x^2 + y^2 + 6x - 10y + 30 = 0$.
STATEMENT - 1 **:** If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C.
and
STATEMENT - 2 **:** If line L_1 is a chord of circle C, then line L_2 is not chord of circle C.
(A) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1.
(B) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a not a correct explanation for STATEMENT - 1.
(C) STATEMENT - 1 is True, STATEMENT - 2 is False.
(D) STATEMENT - 1 is False, STATEMENT - 2 is True.

13. **(C)** Centre $(-3, 5)$, radius = 2
$| p + 6 | < 2\sqrt{13}$
$\Rightarrow \quad -2\sqrt{13} < p + 6 < 2\sqrt{13}$
For second line to be diameter, $| p + 12 | = 0$.

SECTION - III (Linked Comprehension Type)

Paragraph for Question Nos. 14 to 16

Consider the lines

$$L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$$

$$L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}.$$

14. The unit vector perpendicular to both L_1 and L_2 is

(A) $\dfrac{-\vec{i} + 7\vec{j} + 7\vec{k}}{\sqrt{99}}$ (B) $\dfrac{-\vec{i} - 7\vec{j} + 5\vec{k}}{5\sqrt{3}}$

(C) $\dfrac{-\vec{i} + 7\vec{j} + 5\vec{k}}{5\sqrt{3}}$ (D) $\dfrac{7\vec{i} - 7\vec{j} - \vec{k}}{\sqrt{99}}$

14. **(B)** $\begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -i - 7j + 5k$.

15. The shortest distance between L_1 and L_2 is

(A) 0 (B) $\dfrac{17}{\sqrt{3}}$ (C) $\dfrac{41}{5\sqrt{3}}$ (D) $\dfrac{17}{5\sqrt{3}}$

15. **(D)** $((-i - 7j + 5k) / 5\sqrt{3})\ .\ (3i + 4k)$
$= |\ (-3 + 20) / 5\sqrt{3}\ |$.

16. The distance of the point (1, 1, 1) from the plane passing through the point (– 1, 2, – 1) and whose normal is perpendicular to both the lines L_1 and L_2 is

(A) $\dfrac{2}{\sqrt{75}}$ (B) $\dfrac{7}{\sqrt{75}}$ (C) $\dfrac{13}{\sqrt{75}}$ (D) $\dfrac{23}{\sqrt{75}}$

16. **(C)** $-1\,(x + 1) + -7\,(y + 2) + 5\,(z + 1) = 0$
$P : -x - 7y + 5z - 10 \quad \Rightarrow \quad d = |-13 / 5\sqrt{3}\ |$.

Paragraph for Question Nos. 17 to 19

Consider the function $f : (-\infty, \infty) \rightarrow (-\infty, \infty)$ defined by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, \ 0 < a < 2.$$

17. Which of the following is true ?

(A) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

(B) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$

(C) $f'(1) f'(-1) = (2 - a)^2$

(D) $f'(1) f'(-1) = -(2 + a)^2$

18. Which of the following is true ?
(A) $f(x)$ is decreasing on (– 1, 1) and has a local minimum at $x = 1$
(B) $f(x)$ is increasing on (– 1, 1) and has a local maximum at $x = 1$
(C) $f(x)$ is increasing on (– 1, 1) but has neither a local maximum nor a local minimum at $x = 1$
(D) $f(x)$ is decreasing on (– 1, 1) but has neither a local maximum nor a local minimum at $x = 1$

19. Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2} dt$.
Which of the following is true ?
(A) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$
(B) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
(C) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$
(D) $g'(x)$ does not change sign on $(-\infty, \infty)$

Solution for 17 to 19

$$f'(x) = \frac{2a(x^2-1)}{(x^2+ax+1)^2} \qquad f''(1) = \frac{4a}{(a+2)^2} \qquad f''(-1) = \frac{-4a}{(2-a)^2}$$

17. **(A)**
18. **(A)**
19. **(B)** $g'(x) = \frac{f'(e^x).e^x}{1+e^{2x}}$.

SECTION - IV (Matrix Match Type)

20. Consider all possible permutations of the letters of the word ENDEANOEL.
Match the Statements / Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropirate bubbles in the 4 × 4 matrix given in the ORS.

Column I		Column II	
(A)	The number of permutations containing the word ENDEA is	(P)	$5!$
(B)	The number of permutations in which the letter E occurs in the first and the last positions is	(Q)	$2 \times 5!$
(C)	The number of permutations in which none of letters D, L, N occurs in the last five positions is	(R)	$7 \times 5!$
(D)	The number of permutations in which the letters A, E, O occur only in odd positions is	(S)	$21 \times 5!$

20. **(A) → (P)** E → 3, N → 2, D → 1, A → 1, O → 1, L → 1
(B) → (S) $7!/2!$
(C) → (Q) $(4!/2!) \times (5!/3!)$
(D) → (Q) $(5!/3!) \times (4!/2!)$

21. Match the Statements / Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropirate bubbles in the 4 × 4 matrix given in the ORS.

Column I		Column II	
(A)	The minimum value of $\dfrac{x^2+2x+4}{x+2}$ is	(P)	0
(B)	Let A and B be 3 × 3 matrices of real numbers, where A is symmetric, B is skew- symmetric, and $(A + B)(A - B) = (A - B)(A + B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible value of k are	(Q)	1
(C)	Let $a = \log_3 \log_2 2$. An integer k satisfying $1 < 2^{(-k+3-a)} < 2$, must be less than	(R)	2
(D)	If $\sin\theta = \cos\phi$, then the possible values of $\dfrac{1}{\pi}\left(\theta \pm \phi - \dfrac{\pi}{2}\right)$ are	(S)	3

21. **(A) → (R)** $x + (4 / (x + 2))$

$(x + 2) + (4 / (x + 2)) - 2 \geq 2\sqrt{4} - 2$

(B) → (QS) $A \,.\, B = B \,.\, A$

$(AB)^{T} = B^{T} A^{T}$

(C) → (RS) $3^{a} = \log_{3}{}^{2}$

$3^{t} = 2 \quad (t = 3^{a})$

$3 = 2^{1/t} = 2^{3^{-a}} \quad \therefore \quad (1 / 3) < 2^{-k} < (2 / 3) \quad \Rightarrow \quad k = 1$

(D) → (PR) $\theta = (2n + 1)(\pi / 2) \pm \phi$

22. Consider the lines given by

$L_1 : x + 3y - 5 = 0$

$L_2 : 3x - ky - 1 = 0,$

$L_3 : 5x + 2y - 12 = 0$

Match the Statements / Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropirate bubbles in the 4 × 4 matrix given in the ORS.

Column I	Column II
(A) L_1, L_2, L_3 are concurrent, if	(P) $k = -9$
(B) One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(Q) $k = -6/5$
(C) L_1, L_2, L_3 form a triangle, if	(R) $k = 5/6$
(D) L_1, L_2, L_3 do not form a triangle, if	(S) $k = 5$

22. $x + 3y = 5$

$5x + 2y = 12$

$\Rightarrow \quad 13y = 13 \quad \Rightarrow \quad y = 1, x = 2.$

(A) → (S) $6 - k - 1 = 0 \quad \Rightarrow \quad k = 5$

(B) → (PQ) $3 / k = -(1 / 3) \quad \Rightarrow \quad k = -9$

or $\quad 3 / k = -(5 / 2) \quad \Rightarrow \quad k = -6 / 5.$

(C) → (R)

(D) → (PQS)